Transgression on Hyperkähler Manifolds and Generalized Higher Torsion Forms.

A.Gerasimov, A.Kotov

ITEP, B. Cheriomushkinskaya 25, Moscow, 117259 Russia

Transgression of the characteristic classes taking values in the differential forms is a reach source of the interesting algebraic objects. The examples include Chern-Simons and Bott-Chern forms which are given by the transgression of the Chern character form. Chern-Simons forms are defined for a vector bundle over an arbitrary real manifold and are connected with the representation of combinations of Chern classes by the exact form $\omega = d\phi$. Bott-Chern forms are defined for holomorphic hermitian vector bundles over Kähler manifolds. These additional structures allow to use the double transgression $\omega = \partial\overline{\partial}\phi$ to define this invariant. Basically the existence of this representation is a consequence of the action of the multiplicative group of complex numbers $\mathbb{C}^*$ on the cohomology of an arbitrary Kähler manifold.

It is natural to guess that in the case when there is a bigger group acting on the cohomology one should look for more involved objects associated with vector bundles. In this paper we consider the case of the action of the multiplicative group of quaternions $\mathbb{H}^*$ on the cotangent bundle which induces the action of $\mathbb{H}^*$ on the cohomology of the manifold. Supplying the manifold with a metric compatible with the action of $\mathbb{H}^*$ we get a hyperkähler manifold. We propose a new invariant of a hyperholomorphic bundle over a hyperkähler manifold connected with the Chern character form by the *fourth* order "transgression" $\omega = dd_I d_J d_K \phi$. It takes values in differential forms and its zero degree part is hyprholomorphic analog of the logarithm of the holomorphic torsion (holomorphic torsion is trivial for hyperkähler manifolds). This new hypertorsion seems first have appeared in the physical literature [5].

The expression for the hypertorsion in terms of the integration over quaternionic projective plane proposed in this paper is a direct generalization of the formula for the double transgression [11]. The double transgression of the Chern character form in terms of the integration over complex projective plane provides the first example of the series of the regulator maps in algebraic K-theory. We believe that the results of this paper imply (among other interesting applications) that there is a generalization of the regulator maps in algebraic K-theory with the basic simplex being the configuration of linear subspaces in the quaternionic linear spaces.

The paper is organized as follows. In the first part we propose the generalization of the Hodge dd_c-lemma for compact hyperkähler manifolds. This leads to the fourth order transgression of the differential forms. In the second part we consider the transgression of the Chern classes of hyperholomorphic bundles. Application of the results of the first section gives the global construction of the fourth order transgression of the Chern character of hyperholomorphic bundles. Then we give the explicit local construction of this new invariant for the important example of the infinite dimensional bundle arising in the discussion of the local families index theorem. We

define the higher analytic hypertorsion for families of hyperholomorphic bundles on compact hyperkähler manifolds. An explicit formula for the zero-degree part is given in terms of the Laplace operators acting on sections of the vector bundle twisted by the bundle of the differential forms.

Generalization of dd_c-lemma for hyperkähler manifolds

To put the result of this section in the right perspective we begin with the well-known cases of the Riemannian and Kähler manifolds and then consider the case of the Hyperkähler manifold.

Let us given a one-dimensional family of the closed differential forms ω on the compact Riemannian manifold M with the constant image in the de Rham cohomology group.

$$\omega(t) \in \Omega^{closed} \tag{1}$$

$$[\delta_t \omega(t)] = 0 \ \text{ in } \ H^\bullet(M) \tag{2}$$

It implies that the variation of the differential form is exact:

$$\delta_t \omega(t) = d\phi \tag{3}$$

In the presence of the metric on the manifold M one could chose the unique representative for the form ϕ. Let d be the exterior derivative operator, d^* be a conjugated operator with respect to the natural scalar product on the differential forms defined by the metric. Then the corresponding Laplace operator $\Delta = dd^* + d^*d$ and its Green function G are defined:

$$1 = \mathcal{H} + \Delta G \tag{4}$$

where $\mathcal{H}$ is the projector on the harmonic forms. Using the standard considerations [12] one gets the following explicit expression for the form ϕ.

$$\phi = d^* G \delta_t \omega(t) \tag{5}$$

Consider now the case of the Kähler manifold M. Let I be an automorphism of the cotangent bundle corresponding to integrable covariantly constant complex structure on M. One could extended the action of I on $k-$forms for arbitrary k as:

$$ad_I(\alpha_1 \wedge ... \wedge \alpha_k) = \sum_i \alpha_1 ... \wedge I(\alpha_i) \wedge ...\alpha_k. \tag{6}$$

Definition 1 *The Lie algebra $\mathfrak{g}_M$ generated by ad_I is called an isotropy algebra, and the corresponding Lie group $G_I = U(1)$ is called an isotropy group.*

Considering I as an element of the isotropy group G_M we have

$$I(\alpha_1 \wedge ... \wedge \alpha_k) = I(\alpha_1)... \wedge I(\alpha_i) \wedge ...I(\alpha_k) \tag{7}$$

Given a complex structure I, one can introduce the differential operator $d_I = [ad_I, d]$. Since $ad_I(\omega) = i(p-q)\omega$ for ω of type (p,q) we have $d_I = i(\partial_I - \bar{\partial}_I)$. Considering I as an element of the group $(I(\omega) = i^{p-q}\omega)$, the new differential may be represented as $d_I = IdI^{-1}$. These two equivalent representations of d_I immediately imply

$$d_I^2 = dd_I + d_Id = 0. \tag{8}$$

It is useful for further generalizations to introduce the differential operator $d_x = x^0 d + x^1 d_I$ parameterized by the point of the complex plane $(x^0 + x^1 i) \in \mathbb{C}$. We also define the operator $\widehat{x} = x^0 N + x^1 ad_I$ where N is the grading operator acting as k on the differential k-form. These operators have the following obvious properties:

$$\begin{align}
(i) \quad [ad_{\widehat{x}}, d_y] &= d_{xy} \tag{9}\\
(ii) \quad \{d_x, d_y\} &= d_xd_y + d_yd_x = 0 \tag{10}\\
(iii) \quad \{d_x, d_y^*\} &= d_xd_y^* + d_y^*d_x = Re(\bar{x}y)\Delta \tag{11}
\end{align}$$

Let us given a one-dimensional family of the closed $G_M = U(1)$ invariant differential forms $\omega(t)$ on a compact Kähler manifold M. Suppose this family has the constant image in de Rham cohomology of M. Thus we have the conditions:

$$\omega(t) \in \Omega^{closed} \tag{12}$$

$$[\delta_t\omega(t)] = 0 \ \text{ in } \ H^\bullet(M) \tag{13}$$

$$[ad_I, \omega] = 0 \tag{14}$$

In particular the last condition implies that the form ω is d_I-closed and thus the following theorem (Hodge "dd_c-lemma") is applied:

Theorem 1 *[12] Let ω be a d-exact and d_I-closed form on a compact Kähler manifold. Then: $\omega = dd_I\chi$*

Therefore the variation of the form ω is dd_I-exact:

$$\delta_t\omega(t) = dd_I\chi \tag{15}$$

The explicit formula for χ may be given in terms of the Laplace operator and its Green function (see e.g. [12]):

$$\chi = dd^*Gd_Id_I^*G\delta\omega(t) = dd_Id^*d_I^*G^2\delta_t\omega(t)$$

Here we have used the relations (10)(11). Note that the condition (14) may be substituted by a more strong condition on the complex valued differential form ω to be an eigenvalue of the operator ad_I. This gives rise to the same representation (15).

The next case to consider is the differential forms on hyperkähler manifolds.

Definition 2 *[13] A hyperkähler manifold is a Riemannian manifold M with three complex structures I, J and K which satisfy the quaternionic identities $I^2 = J^2 = K^2 = IJK = -1$, such that M is Kähler with respect to any of these structures.*

On a hyperkähler manifold we have a family of integrable complex structures

$$C = \sigma^1 I + \sigma^2 J + \sigma^3 K$$

$$C^2 = -(\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2 = -1$$

parameterized by the points of the sphere S^2, such that M is Kähler with respect to any C.
Obviously $\mathfrak{g}_M = \mathfrak{su}(2)$, $G_M = SU(2) = Sp(1)$. This means that

$$[ad_I, ad_J] = 2ad_K, \quad [ad_J, ad_K] = 2ad_I, \quad [ad_K, ad_I] = 2ad_J.$$

Consider the differential operator $d_x = x^0 d + x^1 d_I + x^2 d_J + x^3 d_K$ parameterized by the points on the quaternioinic plane $x = x^0 + x^1 i + x^2 j + x^3 k \in \mathbb{H}$ and the operator $\widehat{x} = x^0 N + x^1 ad_I + x^2 ad_J + x^3 adK$ acting on the differential forms. These operators satisfy the following relations:

Proposition 1

$$(i) \quad [ad_{\widehat{x}}, d_y] \quad = d_{xy} \tag{16}$$

$$(ii) \quad \{d_x, d_y\} \quad = d_x d_y + d_y d_x = 0 \tag{17}$$

$$(iii) \quad \{d_x, d_y^*\} \quad = d_x d_y^* + d_y^* d_x = Re(\bar{x}y)\Delta, \tag{18}$$

Proof. (i) First let us prove that $[ad_{\widehat{x}}, d] = d_x$. Since the Levi-Chivita connection on M is torsion-free, one has the relation $d(\alpha) = d\xi^i \wedge \nabla_i(\alpha)$, where ξ^i are local coordinates. Operators N, I, J, K are covariantly constant and we get

$$[ad_{\widehat{x}}, d] = (x^0 + x^1 I + x^2 J + x^3 K)(d\xi^i) \wedge \nabla_i =$$

$$= x^0 d + x^1 [ad_I, d] + x^2 [ad_J, d] + x^3 [ad_K, d] = d_x$$

Now we have

$$[ad_{\widehat{x}}, d_y] = (x^1 I + x^2 J + x^3 K)(y^0 + y^1 I + y^2 J + y^3 K)(d\xi^i) \wedge \nabla_i = [ad_{\widehat{xy}}, d] = d_{xy}$$

Using these identities, one can infer that the action of the invertable quaternion U by the conjugation on d_x has a simple form:

$$U d_x U^{-1} = d_{Ux} \tag{19}$$

(ii) Obviously $\{d_x, d_y\}$ is conjugated to $\{d, d_{\bar{x}y}\}$ and therefore (ii) is the consequence of (8).(iii) Since the action of the isotropy group G_M is unitary one has

$$U d_x^* U^{-1} = d_{Ux}^*.$$

Let $L^C = \omega_C \wedge$ be the Hodge operator of the multiplication on the Kähler form for the complex structure C and $\Lambda_C = L_C^*$ be its conjugate. Kodaira's identities

$$d_C^* = [\Lambda_C, d], \quad d^* = -[\Lambda_C, d_C],$$

4

imply the following relations:

$$\{d, d_C^*\} = 0, \quad \{d^*, d_C\} = 0$$

$$\Delta_d = \Delta_{d_C}, \quad \Delta_{d_C} = d_C d_C^* + d_C^* d_C$$

From these identities and the formula for the conjugation of d_x and d_x^*, we easily derive:

$$\{d_x, d_y^*\} = U\{d, d_{\bar{x}y}^*\}U^{-1} = Re(\bar{x}y)U\Delta U^{-1},$$

where $U = |x|^{-1}x$. Finally (iii) follows from the fact that Δ is G_M−invariant. ∎

Consider a one-dimensional family of the closed $G_M = Sp(1)$-invariant differential forms $\omega(t)$ on a compact hyperkähler manifold M and suppose $\omega(t)$ has the constant image in the de Rham cohomology:

$$\omega(t) \in \Omega^{closed} \tag{20}$$

$$[\delta_t \omega(t)] = 0 \ \text{ in } \ H^\bullet(M) \tag{21}$$

$$[ad_{G_M}, \omega] = 0 \tag{22}$$

The last condition implies that $\omega(t)$ is d_C-closed for any C. We would like to show that the variation of the differential form admits the following representation:

$$\delta_t \omega = dd_I d_J d_K(\tau) \tag{23}$$

The variation of the form is exact and the result follows from the proposition:

Proposition 2 *Let ω be an d-exact and d_C-closed differential form of order k for any compatible complex structure C on the compact hyperkähler manifold M. Then there exists a form τ of order $k-4$ such that $\omega = dd_I d_J d_K(\tau)$.*

Proof. Note that if ω is exact and d_C-closed then by Hodge theorem ω is d_C−exact for all C. In particular ω is of type (p,p) with respect to any C from the hyperkähler family.

Let G be the Green operator associated with the Laplacian Δ. Then by taking into account that ω is d_C−exact we obtain

$$\omega = d_C d_C^* G\omega.$$

From the relations (16)-(18) we have

$$\omega = (\prod_{C=I,J,K} d_C d_C^* G)\omega = dd_I d_J d_K(d^* d_I^* d_J^* d_K^* G^4 \omega).$$

Therefore taking $\tau = d^* d_I^* d_J^* d_K^* G^4 \omega$ we immediately obtain the desired formula $\omega = dd_I d_J d_K(\tau)$. ∎

For a four dimensional manifold this relation may be further simplified.

Proposition 3 *Let M be a hyperkähler manifold of dimension 4 and let φ be a smooth function with a compact support. Then $dd_I d_J d_K(\varphi) = 16vol_M \Delta^2(\varphi)$.*

Proof.

Taking into account the properties of the differentials and Kähler forms under conjugation:

$$\begin{cases} d_J &=& JdJ^{-1} \\ d_K &=& -Jd_IJ^{-1} \end{cases}, \quad \begin{cases} J\Lambda_I J^{-1} &=& -\Lambda_I \\ JL_I J^{-1} &=& -L_I \end{cases} \tag{24}$$

we have the following generalized Kodaira identities:

$$\begin{cases} d^* &=& -[\Lambda_C, d_C] \\ d_C^* &=& [\Lambda_C, d] \end{cases}, \quad \begin{cases} d &=& [L_C, d_C^*] \\ d_C &=& -[L_C, d^*] \end{cases}, \tag{25}$$

$$\begin{cases} d_K^* &=& [\Lambda_I, d_J] \\ d_J^* &=& [d_K, \Lambda_I] \end{cases} \tag{26}$$

Let φ be a function with a compact support. With the help of (25),(26) we easily derive the relation

$$(\Lambda_I)^2 dd_I d_J d_K(\varphi) = 2d_I^* d_I d_K^* d_K(\varphi) = 2\Delta^2(\varphi) \tag{27}$$

For an arbitrary top degree differential form ψ on a four dimensional Kähler manifold M there is a simple relation: $(\Lambda_C^2 \psi)\, vol_M = 2\psi$ where vol_M is the volume form on M. Therefore we have the formula:

$$dd_I d_J d_K(\varphi) = vol_M \Delta^2(\varphi).$$

∎

Higher analytic hypertorsion forms

The conditions on the differential forms discussed in the previous section naturally arise when the characteristic classes of vector bundles are considered. Suppose we have a vector bundle $\mathcal{E}$ over the Riemannian manifold M. According to Chern-Weil theory the choice of the connection on the bundle allows to construct the Chern character with values in the closed differential forms. The image in the cohomology lies in the integer lattice $H^{even}(M, \mathbb{Z})$ and gives the topological invariant of the bundle. The smooth deformations of the bundle do not change the cohomology class of the corresponding differential form and the conditions (1),(2) are satisfied. The exactness of the variation of the Chern form allows to construct Chern-Simons differential forms. For instance considering the second Chern class $c_2 = -\frac{1}{8\pi^2} TrF \wedge F$ for a one dimensional family of the connections on the bundle parameterized by the variable t we get an example of the Chern-Simons form $CS(A) = Tr(A \wedge dA + \frac{2}{3}A^3)$ through the relation:

$$\delta_t c_2(A) = -\frac{1}{8\pi^2} d\delta_t CS(A) \tag{28}$$

In the case of holomorphic bundles on the Kähler manifold the choice of a hermitian metric on the bundle leads to the Chern character form subjected to the additional condition. The corresponding cohomology classe should be invariant under the natural action of $U(1)$ on the cohomology of the Kähler manifolds. Thus we have all the conditions (12),(13),(14) satisfied and this allows to define Bott-Chern differential forms (i.e. see [7]for the detailed discussion) .

6

The next interesting case is a hyperholomorphic bundle on the hyperkähler manifold. Hyperholomorphic bundle is a hermitian bundle vector bundle which is holomorphic with respect to any of the compatible holomorphic structures associated with the hyperkähler manifold. The corresponding characteristic classes are subjected to the condition to be invariant with respect to the action of the isotropy group on the cohomology [17]. This provides additional condition (22) and allows to apply the generalization of the dd_c-lemma from the first part of the paper. Thus we have derived the existence of the fourth order transgression of the Chern character form of an arbitrary hyperholomorphic vector bundle. Note however that this arguments is global and one could wonder if there exists a simple local expression for the resulted differential form.

Below we give the explicit answer for one particular interesting example. We consider the infinite dimensional hyperholomorphic bundles naturally arising from the families of the hyperkähler manifolds supplied with a finite dimensional hyperholomorphic bundle. We provide local construction of the fourth order transgression in this case and give the explicit formula for the resulted resulted generalized higher torsion form (hypertorsion form). Local construction for the general case of an arbitrary hyperholomorphic bundle will be discussed elsewhere.

Consider the local universal family $\pi : M \times B \to B$ of the deformations of a hyperholomorphic bundle $\mathcal{V}$ with a hermitian metric on the fiber M parameterized by B. Let $\mathcal{W}$ be a corresponding universal bundle over $M \times B$. The family of the Dirac operators $D = D^- + D^+$ acting along the fiber M on the twisted spinor bundles $\mathcal{E} = \mathcal{V} \otimes \mathcal{S}(M) = \mathcal{V} \otimes \mathcal{S}(M)_+ \oplus \mathcal{V} \otimes \mathcal{S}(M)_-$ defines the virtual index bundle $\widetilde{\mathcal{W}} \equiv Ind(D^+)$ on the base of the fibration. This provides two closed differential forms on the base B. The first form is the product of the Chern character of $\mathcal{W}$ and $\widehat{A}$ class of the tangent bundle to M integrated over the fiber of the projection. The other one is the product of the Chern class of $\widetilde{\mathcal{W}}$ supplied with the L_2 metric. The local families index theorem of Atiyah and Singer [1] claims that:

$$ch(\widetilde{\mathcal{W}}) = \pi_*[ch(\mathcal{W})\widehat{A}(TM)] \text{ in } H^{even}(B, \mathbb{Q}) \tag{29}$$

One can construct a one-dimensional family of Quillen superconnections acting in the associated infinite dimensional hyperholomorphic bundle $\widehat{\mathcal{W}}$ of twisted spinor sections $\Gamma(\mathcal{E}|_M, M)$ over B. This gives rise to the representative of the Chern character in the differential forms interpolating between the l.h.s. and r.h.s. of (29). Locally over B both parts of (29) are given by exact forms and by the general properties of the Chern classes of hyperholomorphic bundles [17] are $G_M = Sp(1)$-invarinat. We derive the explicit formula for fourth order transgression of their difference. This defines hypertorsion differential form for the families of hyperholomorphic bundles. Let us start with short description of Quillen superconnection formalism. Consider $\mathbb{Z}_2-$graded vector bundle $\mathcal{E}$ and let τ be the operator defining the $\mathbb{Z}_2$-grading on $\mathcal{E}$ i.e. $\tau = \pm 1$ on $\mathcal{E}_\pm$. The algebra $End(\mathcal{E})$ is naturally $\mathbb{Z}_2$ graded algebra. We set a $\mathbb{Z}_2-$grading to the bundle of $\mathcal{E}$-valued differential forms $\Lambda^*(\mathcal{E}, M)$ as a graded tensor product. For $A, A' \in \Lambda^*(\mathcal{E}, M)$ the supercommutator $[A, A']$ is given by:

$$[A, A'] = AA' - (-1)^{deg(A)\,deg(A')}A'A \tag{30}$$

$\mathbb{Z}_2$-grading allows to define supertrace $Str(A)$ as:

$$Str(A) = Tr(\tau A) \tag{31}$$

with the property to be zero on supercommutators.

The form $Str(e^{-\nabla^2})$ is a closed differential form representing Chern character of the virtual bundle $\mathcal{E}_+ \ominus \mathcal{E}_-$:

$$ch(\mathcal{E}_+ \ominus \mathcal{E}_-) = ch(\mathcal{E}_+) - ch(\mathcal{E}_-) = Str(e^{-\nabla^2}) \tag{32}$$

Thus defined Chern classes differ from the standard Chern classes by the multiplication of the degree $2k$ components by $(2\pi i)^k$. In the following we will always this normalization.

Let us given an odd self adjoint operator V acting on $\mathcal{E}$ (i.e odd section of $End(\mathcal{E})$). We could combine it with the connection to get the Quillen superconnection on $\mathbb{Z}_2$-bundle $\mathcal{E}$.

Definition 3 *A differential operator* $\mathbb{A} : \Lambda^*(\mathcal{E}, M) \rightarrow \Lambda^*(\mathcal{E}, M)$ *of order* 1 *with respect to* $\mathbb{Z}_2-$*grading is a Quillen's superconnection if*

$$\mathbb{A}(\omega s) = (d\omega)s + (-1)^{deg\,\omega}\omega\mathbb{A}(s)$$

In fact, we could construct a family of the superconnections depending on a real positive parameter $t \in \mathbb{R}_+$:

$$\mathbb{A}_t = \nabla + \sqrt{t}V \tag{33}$$

Here ∇ and V are even and odd parts of the superconnection.

The space B of local deformations of a hyperholomorphic bundle $\mathcal{V}$ on M is naturally supplied with a hyperkähler structure. We show that Quillen superconnections defined over the base B and extended base $B \times \mathbb{H}$ are hyperholomorphic.

Proposition 4 *Let B be a space of local deformations of a hyperholomorphic vector bundle with a hermitian metric over a hyperkähler manifold M.*

(i) The superconnection $\mathbb{A}_t = d^B + \sqrt{t}D$ is hyperholomorphic over B.

(ii) Consider the operator $D_x = x^0 D + x^1 D_I + x^2 D_J + x^3 D_K$, where $D_L = c(L)Dc(L)^{-1}$. Chose a hyperkähler structure on quaternionic plane $x = x^0 + x^1 i + x^2 j + x^3 k \in \mathbb{H}$ by considering the right multiplication by quaternionic units $-i, -j, -k$.

Then the superconnection $\mathbb{A}_x = d^{\mathbb{H}} + d^B + D_x$ is hyperholomorphic over $B \times \mathbb{H}$.

Proof.

(i) Let C be a complex structure compatible with the hyperkähler structure on $B \times M$. We will use the results from the end of the Appendix B. The isomorphism $\mathcal{E} = \mathcal{V} \otimes \mathcal{S} \cong \Lambda^{(*,0)}(\mathcal{V}, M)$, $D = \sqrt{2}(\nabla'_C + (\nabla'_C)^*)$ on M leads to the decomposition $D = D' + D''$ where $D' = \sqrt{2}\nabla'_C$,$D'' = \sqrt{2}(\nabla'_C)^*$ have the types $(1,0)$ and $(0,1)$. Using the variant of the Kadaira identity: $\{\overline{\partial}_C^B, (\nabla_C)^*\} = 0$ we have:

$$(\mathbb{A}_t'')^2 = (\overline{\partial}_C^B + \sqrt{2t}(\nabla_C)^*)^2 = 0 \tag{34}$$

In particular for $\mathbb{A}_{t,C} = i(\mathbb{A}_t' - \mathbb{A}_t'')$ the following identities holds:

$$\mathbb{A}_{t,I}^2 = \mathbb{A}_t^2 \tag{35}$$

Note that here C acts on the total tangent bundle to $B \times M$. Taking into account that (34) holds for any C we infer that $\mathbb{A}_t$ is hyperholomorphic over B.

(ii) Note that $[ad_\phi, D_x] = D_{\phi x}$, where $\phi = \phi^1 I + \phi^2 J + \phi^3 K$ is an arbitrary generator of $Sp(1)-$action on twisted spinors. Let us start with the complex structure defined by I.

Then $\sqrt{t}D'$ is gauge equivalent to $D'_x = z_1 D' + \bar{z}_2 D'_J$, and $\sqrt{t}D''$ is gauge equivalent to $D''_x = \bar{z}_1 D'' + z_2 D''_J$, where

$$z_1 = x^0 + \sqrt{-1}x^1, \quad z_2 = x^2 + \sqrt{-1}x^3$$

$$t = |x|^2 = (x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2$$

and $D'_J = JD'J^{-1}$ $D''_J = JD''J^{-1}$. The connection operators $\partial^B + \sqrt{t}D'$ and $\bar{\partial}^B + \sqrt{t}D''$ are gauge equivalent to $\partial^B + D'_x$ and $\bar{\partial}^B + D''_x$. Therefore their squares are equal to zero.

Let us decompose $\mathbb{A}_x = \mathbb{A}' + \mathbb{A}''$, where

$$\mathbb{A}' = d\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + dz_2 \frac{\partial}{\partial z_2} + \partial^B + z_1 D' + \bar{z}_2 D'_J$$

$$\mathbb{A}'' = dz_1 \frac{\partial}{\partial z_1} + d\bar{z}_2 \frac{\partial}{\partial \bar{z}_2} + \bar{\partial}^B + \bar{z}_1 D'' + z_2 D''_J$$

It is clear that $(\mathbb{A}')^2 = (\mathbb{A}'')^2 = 0$

Since $(\bar{z}_1, z_2)$ are holomorphic coordinates on $\mathbb{H}$ with respect to right multiplication by $-i$ the superconnection $\mathbb{A}$ is holomorphic on $B \times \mathbb{H}$. The same arguments work for any compatible complex structure C. Taking into account the isomorphism $\mathcal{S} \otimes \mathcal{V} \simeq \Lambda_C^{(*,0)}(\mathcal{V}, M)$ for any complex structure C we conclude that $\mathbb{A}_x$ is hyperholomorphic over $B \times \mathbb{H}$. ∎

Consider the Chern character form defined by superconnection $\mathbb{A}_t$ (see [6], [4], [7]):

$$ch(\mathbb{A}_t) = Str e^{-\mathbb{A}_t^2} \tag{36}$$

It interpolates between Chern character form $ch(\widetilde{\mathcal{W}})$ for the L_2-metric on the index bundle $\widetilde{\mathcal{W}}$ and characteristic class $ch(\mathcal{W})\widehat{A}(TM)$ integrated along the fiber:

$$ch(\widetilde{\mathcal{W}}) = \lim_{t \to \infty} Str e^{-\mathbb{A}_t^2} \tag{37}$$

$$\int_M ch(\mathcal{W})\widehat{A}(TM) = \lim_{t \to 0} Str e^{-\mathbb{A}_t^2} \tag{38}$$

where $ch(\mathcal{W})$ is Chern character form of the canonical connection on the bundle $\mathcal{W}$ over $M \times B$ and $\widehat{A}(TM)$ is the multiplicative genus given by the power series:

$$\widehat{A}(x) = \frac{\frac{x}{2}}{sinh(\frac{x}{2})}$$

of the curvature of the Levi-Chivita connection over M.

Theorem 2 *The following transgression formula holds:*

$$ch(\widetilde{\mathcal{W}}) - \int_M ch(\mathcal{W})\widehat{A}(TM) = \frac{1}{24}d^B d_I^B d_J^B d_K^B \beta \tag{39}$$

where

$$\beta = \sum_{C=I,J,K} \int_{\to 0}^{+\infty} Str(\int_0^1 ad_C e^{-\tau \mathbb{A}_t^2} ad_C e^{-(1-\tau)\mathbb{A}_t^2} d\tau)\frac{dt}{t} \tag{40}$$

9

is a higher hypertorsion differential form. The zero degree part of β

$$\beta_0 = \int\limits_{\to 0}^{+\infty} Str(\sum_{C=I,J,K} (adc_C)^2 e^{-tD^2} \frac{dt}{t})$$

may be expressed in terms of the Laplace operators Δ_q acting on q-froms: $D^2 = \sum_q \Delta_q$ as the logarithm of the hypertorsion T_h:

$$\beta_0 \quad = \quad 3logT_h \tag{41}$$

$$T_h \quad = \quad \prod_{q=0}^{2k} (det'\Delta_q)^{(-1)^q q^2} \tag{42}$$

The definition of the "regularized" integral $\int\limits_{\to 0}^{\infty} f(t)\frac{dt}{t}$ is given in Appendix A.

First we give a simple local argument in favor of the existence of the fourth order transgression and then give the formal proof of the theorem.

Taking into account the identity which follows from Proposition 3 :

$$d^{\mathbb{H}} d_I^{\mathbb{H}} d_J^{\mathbb{H}} d_K^{\mathbb{H}} \ln|x|^2 = 16\pi^2(\delta_\infty - \delta_0) \tag{43}$$

we have the following representation:

$$Stre^{-\mathbb{A}_t^2}|_0^{+\infty} = \frac{1}{16\pi^2} \int\limits_{\mathbb{H}} Stre^{-\mathbb{A}_x^2} d^{\mathbb{H}} d_I^{\mathbb{H}} d_J^{\mathbb{H}} d_K^{\mathbb{H}} ln|x|^2 \tag{44}$$

Since $\mathbb{A}_x = d^{\mathbb{H}} + d^B + D_x$ is hyperholomorphic over $B \times \mathbb{H}$, we obtain

$$Stre^{-\mathbb{A}_t^2}|_0^{+\infty} = \frac{1}{16\pi^2} d^B d_I^B d_J^B d_K^B \int\limits_{\mathbb{H}} Stre^{-\mathbb{A}_x^2} ln|x|^2 \tag{45}$$

This leads to the fourth-order transgression of the difference of the Chern character forms:

$$ch(\widetilde{\mathcal{W}}) - \int\limits_M ch(\mathcal{W})\widehat{A}(M) = \frac{1}{16\pi^2} d^B d_I^B d_J^B d_K^B \int\limits_{\mathbb{H}} Stre^{-\mathbb{A}_x^2} ln|x|^2 \tag{46}$$

This representation provides the direct generalization of the representation for the higher holomorphic torsion form in terms of the integration over auxiliary complex plane given [11].

One could reduce the expression in r.h.s. to the one given in the Theorem 2. However to make analytic regularization more explicit we proceed with the direct derivation of (39)(40).

Proof of the theorem.

Let us start with the following lemma.

Lemma 1 *We have:*

$$t\partial_t(t\partial_t + 1)Stre^{-\mathbb{A}_t^2} = \frac{1}{8} d^B d_I^B d_J^B d_K^B Str(\int\limits_0^1 ad_I e^{-\tau \mathbb{A}_t^2} ad_I e^{-(1-\tau)\mathbb{A}_t^2} d\tau) \tag{47}$$

Proof.

By simple calculation using the relations $\mathbb{A}_t^2 = \mathbb{A}_{t,I}^2$, $[\mathbb{A}_{t,I}, \mathbb{A}_t^2] = 0$ (see [7] for similar considerations) we get:

$$\partial_t Str e^{-\mathbb{A}_t^2} = -\frac{1}{2\sqrt{t}} Str([\mathbb{A}_t, D]e^{-\mathbb{A}_t^2}) = -d^B \frac{1}{2\sqrt{t}} Str(De^{-\mathbb{A}_t^2}) =$$

$$= -d^B \frac{1}{2t} Str([\mathbb{A}_{t,I}, ad_I]e^{-\mathbb{A}_t^2}) = -\frac{1}{2t} d^B d_I^B Str(ad_I e^{-\mathbb{A}_t^2})$$

Thus we have the relation:

$$t\partial_t Str e^{-\mathbb{A}_t^2} = -\frac{1}{2} d^B d_I^B Str(ad_I e^{-\mathbb{A}_t^2}) \tag{48}$$

Applying the formula of differentiation:

$$\frac{d}{dt} e^{-A(t)} = -\int_0^1 e^{-\tau A(t)} \frac{d}{dt} A(t) e^{-(1-\tau)A(t)} d\tau \tag{49}$$

we derive

$$\partial_t Str(ad_I e^{-\mathbb{A}_t^2}) = -\frac{1}{2\sqrt{t}} Str(ad_I \int_0^1 e^{-\tau \mathbb{A}_{t,J}^2}[\mathbb{A}_{t,J}, D_J] e^{-(1-\tau)\mathbb{A}_{t,J}^2} d\tau) =$$

$$= \frac{1}{2\sqrt{t}} (-d_J^B Str(ad_I \int_0^1 e^{-\tau \mathbb{A}_t^2} D_J e^{-(1-\tau)\mathbb{A}_t^2} d\tau) + Str([\mathbb{A}_{t,J}, ad_I] \int_0^1 e^{-\tau \mathbb{A}_t^2} D_J e^{-(1-\tau)\mathbb{A}_t^2} d\tau)) =$$

$$= -\frac{1}{2t} d_J^B Str(ad_I \int_0^1 e^{-\tau \mathbb{A}_t^2}[\mathbb{A}_{t,K}, ad_I] e^{-(1-\tau)\mathbb{A}_t^2} d\tau) - \frac{1}{2} Str(D_K \int_0^1 e^{-\tau \mathbb{A}_t^2} D_J e^{-(1-\tau)\mathbb{A}_t^2} d\tau)$$

The first part of the expression is equal to $-\frac{1}{4t} d_J^B d_K^B Str(\int_0^1 ad_I e^{-\tau \mathbb{A}_t^2} ad_I e^{-(1-\tau)\mathbb{A}_t^2} d\tau)$. Acting by $t\partial_t$ on (48) and using the result of the previous calculation we get

$$(t\partial_t)^2 Str e^{-\mathbb{A}_t^2} = \frac{1}{8} d^B d_I^B d_J^B d_K^B Str(\int_0^1 ad_I e^{-\tau \mathbb{A}_t^2} ad_I e^{-(1-\tau)\mathbb{A}_t^2} d\tau) + \frac{t}{4} d^B \alpha, \tag{50}$$

where

$$\alpha = d_I^B Str(D_K \int_0^1 e^{-\tau \mathbb{A}_t^2} D_J e^{-(1-\tau)\mathbb{A}_t^2} d\tau)$$

The next step is to obtain α.

$$\alpha = \int_0^1 d\tau (Str([\mathbb{A}_{t,I}, D_K] e^{-\tau \mathbb{A}_t^2} D_J e^{-(1-\tau)\mathbb{A}_t^2} - D_K e^{-\tau \mathbb{A}_t^2}[\mathbb{A}_{t,I}, D_J] e^{-(1-\tau)\mathbb{A}_t^2}))$$

Note that using:

$$[\mathbb{A}_{t,I}, ad_I] = \sqrt{t}D \tag{51}$$

one could get the following relations:

$$[\mathbb{A}_{t,I}, D_K] = \frac{1}{\sqrt{t}}[\mathbb{A}_{t,I}, [\mathbb{A}_{t,I}, ad_J]] = \frac{1}{\sqrt{t}}[\mathbb{A}_t^2, ad_J] \tag{52}$$

$$[\mathbb{A}_{t,I}, D_J] = -\frac{1}{\sqrt{t}}[\mathbb{A}_{t,I}, [\mathbb{A}_{t,I}, ad_K]] = -\frac{1}{\sqrt{t}}[\mathbb{A}_t^2, ad_K] \tag{53}$$

So we have

$$\alpha = \frac{1}{\sqrt{t}}\int_0^1 d\tau Str([\mathbb{A}_t^2, ad_J]e^{-\tau\mathbb{A}_t^2}D_J + D_K e^{-\tau\mathbb{A}_t^2}[\mathbb{A}_t^2, ad_K])e^{-(1-\tau)\mathbb{A}_t^2} =$$

$$= \frac{1}{\sqrt{t}}\int_0^1 d\tau Str(ad_J\partial_\tau e^{-\tau\mathbb{A}_t^2}D_J e^{-(1-\tau)\mathbb{A}_t^2} - D_K\partial_\tau e^{-\tau\mathbb{A}_t^2}ad_K e^{-(1-\tau)\mathbb{A}_t^2}) =$$

$$= \frac{1}{\sqrt{t}}Str(ad_J[e^{-\mathbb{A}_t^2}, D_J] - D_K[e^{-\mathbb{A}_t^2}, ad_K]) = \frac{1}{\sqrt{t}}Str e^{-\mathbb{A}_t^2}([D_J, ad_J] +$$

$$+ [D_K, ad_K]) = \frac{2}{\sqrt{t}}Str D e^{-\mathbb{A}_t^2}.$$

Taking into account (51) we derive:

$$\alpha = \frac{2}{t}d_I^B Str(ad_I e^{-\mathbb{A}_t^2}). \tag{54}$$

After substitution of (54) in (50) we obtain:

$$(t\partial_t)^2 Str e^{-\mathbb{A}_t^2} = \frac{1}{8}d^B d_I^B d_J^B d_K^B \Phi(t) + \frac{1}{2}d^B d_I^B Str(ad_I e^{-\mathbb{A}_t^2}), \tag{55}$$

where

$$\Phi(\kappa) = Str(\int_0^1 ad_I e^{-\tau\mathbb{A}_\kappa^2} ad_I e^{-(1-\tau)\mathbb{A}_\kappa^2} d\tau)$$

By using (48) we immediately prove the lemma. ∎

Let us apply (64) to the combination $G_{\mathbb{A}}(t) = Str e^{-\mathbb{A}_t^2} - ch(\widetilde{\mathcal{W}})$. Note that $G_{\mathbb{A}}(t) = O(t^{-\frac{1}{2}})$ as $t \to \infty$. Thus we have:

$$\int_{\to 0}^{\infty}(t\partial_t(t\partial_t + 1)(Str e^{-\mathbb{A}_t^2} - ch(\widetilde{\mathcal{W}})\frac{dt}{t} =$$

$$= -(t\partial_t + 1)G_{\mathbb{A}}|_{t=0} = ch(\widetilde{\mathcal{W}}) - Str e^{-\mathbb{A}_t^2}(0) \tag{56}$$

Taking into account (47) we have proved the first part of the theorem.

Now let us prove the formula for the zero-degree part of hypertorsion form.

It is clear that

$$\beta_0 = 3 \int\limits_{\to 0}^{+\infty} Str(\int\limits_0^1 ad_I e^{-\tau t D^2} ad_I e^{-(1-\tau)t D^2} d\tau) \frac{dt}{t} = 3 \int\limits_{\to 0}^{+\infty} Str(ad_I^2 e^{-t D^2}) \frac{dt}{t}$$

The following identity from the Appendix A being applied to the trace of the positive self-adjoint operator $\hat{H}$:

$$\int\limits_{\to 0}^{\infty} Tr e^{-t\hat{H}} \frac{dt}{t} = -logdet'\hat{H},$$

immediately leads to the representation of the zero-degree part of the hypertorsion form β_0 in terms of infinite determinants.

Recall that ad_I acts on $(q,0)-$forms on $4k$-dimensional Kähler manifold as $i(q-k)$. So $(ad_I)^2 = -(q-k)^2$ and hence

$$\beta_0 \equiv 3 \log T_h = 3 \sum_{q=0}^{2k} (-1)^q (q-k)^2 Tr\Delta_q^{-s} = 3 \sum_{q=0}^{2k} (-1)^q (q-k)^2 logdet'\Delta_q,$$

where Δ_q is the Laplace operator acting on $\mathcal{W}$ valued $(q,0)-$forms, $\Delta = D^2 = \sum_q \Delta_q$, and $det'\Delta_q$ is the regularalized determinant. We have

$$T_h = \prod_{q=0}^{2k} det'\Delta_q^{(-1)^q (q-k)^2}.$$

The usual analytic torsion for holomorphic bundle over a complex manifold (see [4], [7], [15]) is given by $T = \prod\limits_{q=0}^{2k} (det'\Delta_q)^{q(-1)^q}$. In the following lemma we prove that analytic torsion for a hyperholomorphic bundle over a hyperkähler manifold is trivial.

Lemma 2 *Let M be a hyperkähler manifold and let $\mathcal{V}$ be a hyperholomorphic bundle over M. Then $T = 1$.*

Proof.
Let us write down the expression for $logT$ in the following form:

$$\log T = \sum_{q=0}^{2k} (-1)^q (-\frac{\partial}{\partial s}_{|s=0}) tr(h+k)\Delta_q^{-s},$$

where $h = \frac{1}{i}ad_I = q - k$. Let $S_\lambda^{\pm}$ be the eigen-spaces of $\Delta_{\pm}$, where

$$\Delta_+ = \sum_{q-even} \Delta_q, \quad \Delta_- = \sum_{q-odd} \Delta_q.$$

Since $D\Delta_{\pm} = \Delta_{\mp}D$ then $D : S_\lambda^{\pm} \simeq S_\lambda^{\mp}$. Thus we conclude that

$$\sum_{q=0}^{2k} (-1)^q tr\Delta_q^{-s} = 0.$$

Moreover, the Laplace operator is $Sp(1)-$invariant, so the eigen-subspaces $S_\lambda^\pm$ are $\mathfrak{sp}(1)-$modules. It follows that $tr(h)_{|S_\lambda^\pm} = 0$ and $trh\Delta_+^s = trh\Delta_-^s = 0$. Therefore $\ln T = 0$ and $T = 1$. $\blacksquare$

This lemma implies that

$$T_h = \prod_{q=0}^{2k} det'\Delta_q^{(-1)^q q^2}.$$

$\blacksquare$

There is an interesting particular case of the theorem we have proved. Let $dimM = 4$ and $ch_{[2]}$ be a component of the Chern character taking values in four-forms . In this case $T_h = (det'\Delta_0)^2$ and we have:

$$ch_{[2]}(\widetilde{\mathcal{W}}) = (\int_M ch(\mathcal{W})\widehat{A}(M))_{[2]} + \frac{1}{4}d^B d_I^B d_J^B d_K^B log(det'\Delta_0)$$

If in addition $dimB = 4$ then

$$ch_{[2]}(\widetilde{\mathcal{W}}) = (\int_M ch(\mathcal{W})\widehat{A}(M))_{[2]} + \frac{1}{4}vol_B\Delta_B^2 log(det'\Delta_0)$$

This formula was proposed in the physical literature in [16],[8] for the case of $M = T^4$ and in [9] (see also [3]) for the instantons over $M = \mathbb{R}^4$.

Appendix A: Regularization of integrals.

In this appendix we define the regularization of some class of the integrals using analytic continuation. This regualrization is a standard tool in the theory of higher analytic torsion [6, 7]

Let $G(t)$ be a continuous function defined for $t > 0$ with sufficiently rapid decay as $t \to \infty$. We also assume that it has an asymptotic expansion as $t \to 0$:

$$G(t) = \sum_{i=-n}^{0} G_i t^i + O(t) \tag{57}$$

The following integral:

$$\zeta_G(s) = \frac{1}{\Gamma(s)} \int_0^\infty G(t)t^{s-1}dt \tag{58}$$

converges for $Re(s) > n$ and has the analytic extension to the whole complex plane. We define the value of (in general divergent) integral as follows:

$$\int_{\to 0}^{\infty} G(t)\frac{dt}{t} \equiv \zeta_G'(0) \tag{59}$$

For instance by this definition for $G(t) = e^{-th}$ we have:

$$\int_{\to 0}^{\infty} e^{-th}\frac{dt}{t} = \zeta_{e^{-th}}'(0) = -log(h) \tag{60}$$

Note that thus defined integral has the usual property for the total derivative of a regular function:

$$\int_{\to 0}^{\infty} (t\partial_t F(t))\frac{dt}{t} = -F(0) \tag{61}$$

14

If $F(t)$ has a more general behaviour (57), the value at $t = 0$ of the regular part of $F(t)$ appears in r.h.s of (61) instead of $F(0)$.

We need the following consequence of this property. Consider the regularized integral of the function $H(t) = t\partial_t(t\partial_t + 1)G(t)$ for regular G:

$$\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty (t\partial_t(t\partial_t + 1)G(t))t^{s-1}dt \tag{62}$$

Then the following obvious identity holds:

$$\zeta_H'(0) = -G(0) \tag{63}$$

and we have:

$$\int_{\to 0}^\infty (t\partial_t(t\partial_t + 1)G(t))\frac{dt}{t} = -G(0) \tag{64}$$

Appendix B: Dirac operator on hyperkähler manifolds

Here we recall the interrelation of spin structures and complex structures on the Kähler and hyperkähler manifolds with the emphasis on the properties of the Dirac operator.

Definition 4 *[4] Let V be a real $n-$dimensional vector space with positive quadratic form g. The Clifford algebra of (V, g), denoted by $Cl(V)$, is the algebra over $\mathbb{R}$ generated by V with the relations $xy + yx = -2g(x, y)$. A self-adjoint Hermitian module E of $Cl(V)$ is called a Clifford module.*

Let e^i, $i = 1, ...n$ be an orthogonal basis of V and let c^i be an element of $Cl(V)$ corresponding to e^i. One can extend this map to the isomorphism of graded $O(V)$ modules $c : \Lambda^*(V) \to Cl(V)$ by sending $e^{i_1} \wedge ... \wedge e^{i_k} \mapsto c^{i_1}....c^{i_k}$. The chirality operator $\Gamma = (\sqrt{-1})^{[\frac{n+1}{2}]}c^1...c^n$ satisfies $\Gamma^2 = 1$ and defines a $\mathbb{Z}_2-$grading on $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$. Taking $v \in V$ one can check that $c(v)\Gamma = (-1)^{n+1}\Gamma c(v)$.

The subspace $Cl^2(V) = c(\Lambda^2(V))$ is a Lie subalgebra of $Cl(V)$ which is isomorphic to $\mathfrak{so}(V)$ under the map $\tau : Cl^2(V) \simeq \mathfrak{so}(V)$, $\tau(a)x = [a, x]$, $v \in V$. The group $Spin(V)$ is obtained by exponentiation of the Lie algebra $Cl^2(V)$ inside the Clifford algebra $Cl(V)$.

Let V be a hermitian vector space with a complex structure I. Let $\omega_I = g(I\cdot, \cdot)$ be the corresponding real nondegenerate $2-$form, which can be considered by duality as an element of $\Lambda^2(V)$. Using the isomorphism $Cl^2(V) \simeq \mathfrak{so}(V)$ it is possible to show that $c(\omega^I) = 2I$.

Let us decompose $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$ into holomorphic and antiholomorphic parts $V^c = W \oplus \bar{W}$. Since W and $(\bar{W})$ are isotropic subspaces with respect to the scalar product extended by complex linearity, then $Cl(W)$ and $Cl(\bar{W})$ are commutative graded algebras.

Let us define an irreducible Clifford module, denoted by S, which is called a spin module as a hermitian complex space $\Lambda^*(W)$, provided with the following Clifford action

$$c(w) = \begin{cases} \sqrt{2}\epsilon(w), & w \in W, \\ -\sqrt{2}\iota(\bar{w}), & w \in \bar{W}, \end{cases}$$

where $\epsilon(w)$ is the exterior product of w, and $\iota(\bar{w})$ is the contraction with the hermitian dual covector. The spin representation constructed as above has a unique up to multiplication by

unitary complex numbers normalized vacuum vector $|1\rangle$, which satisfies the following conditions: $Cl(\bar{W})|1\rangle = 0$ and $S = Cl(W)|1\rangle$. The module S descends a natural $\mathbb{Z}$−grading from the space $Cl(W) = \bigoplus_{q=0}^{n} Cl^q(W)$. One can verify that

$$c(\omega^I)_{|Cl^q(W)|1\rangle} = i(2q - n).$$

Let V be a self-adjoint $\mathbb{H}$−module of dimension $4k$. The correspondence between the families of complex structures C and the associated 2−forms ω^C immediately leads to the inclusion $\mathfrak{sp}(1) \hookrightarrow Cl(V)$, $C \to \frac{1}{2}c(\omega^C)$, which can be exponentiated inside $Cl(V)$ to the inclusion $Sp(1) \hookrightarrow Spin(V)$. Therefore the group of unitary quaternions $Sp(1)$ acts on the spin module S, such that $c(x)c(v)c(x)^{-1}s = c(x(v))s$, where $x \in Sp(1)$, $v \in V^c$, $s \in S$. If $dimV = 4$ then the subspace of self-dual 2−forms $\Lambda_+^2(V)$ is spanned by ω^C ([2]).

Since $\mathfrak{sp}(1) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$, one can choose the $\mathfrak{sl}_2$−generators $h = \frac{1}{2i}c(\omega^I)$, $e = \frac{1}{4}(c(\omega^J) - ic(\omega^K))$, $f = -\frac{1}{4}(c(\omega^J) + ic(\omega^K))$ with the relations $[h, e] = -2f$, $[h, f] = -2f$, $[e, f] = h$.

Let us consider the spin module S over $Cl(V)$ as the linear space of $(p, 0)$−forms with respect to I. Then we have the following simple property:

Proposition 5 *The form $\Omega = \frac{1}{4}(\omega^J - i\omega^K)$ is of type $(2, 0)$ with respect to I. Moreover, the operator e acts as the exterior product with Ω and the operator f acts as the contraction with Ω.*

Since $h_{|\Lambda^{q,0}(V)} = q - k$, where $q = 0, ...2k$ we see that the operator h defines $\mathbb{Z}$−grading on S.

Definition 5 *[4] A Clifford module $\mathcal{E}$ over an even dimensional Riemannian manifold M is a $\mathbb{Z}_2$−graded hermitian bundle of Clifford modules $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ over a bundle $Cl(M)$ of Clifford algebras with the unitary connection $\nabla^{\mathcal{E}}$, which is compatible with the Levi-Chivita connection, extended to $Cl(V)$. If $\mathcal{V}$ is a hermitian vector bundle, then $\mathcal{V} \otimes \mathcal{E}$ is the twisted Clifford module with the Clifford action $1 \otimes c(a)$ and with the connection $\nabla^{\mathcal{V} \otimes \mathcal{E}} = \nabla^{\mathcal{V}} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}}$.*

The action of one-forms $\Lambda^1(M)$ on $\mathcal{E}$ defines a $C^\infty(M)$−linear morphism of bundles over M, written as $\Lambda^1(\mathcal{E}_\mp, M) \to \Gamma(\mathcal{E}_\pm, M)$. So one can introduce a generalized Dirac operator, acting as follows:

$$D : \Gamma(\mathcal{E}_\mp, M) \xrightarrow{\nabla^{\mathcal{E}_\mp}} \Lambda^1(\mathcal{E}_\mp, M) \to \Gamma(\mathcal{E}_\pm, M).$$

Let us define a *spin bundle* as a Clifford bundle of spin models. Given an almost complex structure I one can construct a bundle of spin modules. If the Riemannian manifold is Kähler we have a subbundle of $(p, 0)$−forms $\Lambda^{*,0}(M)$. This bundle has the structure of the Clifford module. More generally there is

Proposition 6 *[4] Let $\mathcal{V}$ be a holomorphic vector bundle with hermitian metric on a Kähler manifold. The tensor product of the Levi-Chivita connection on $\Lambda^{*,0}(M)$ with the canonical connection on $\mathcal{V}$ gives Clifford connection on the Clifford module $\Lambda^{*,0}(\mathcal{V}, M)$. Let ∇' be (1,0)-part of the connection on the bundle $\Lambda^{*,0}(\mathcal{V}, M)$. Then the Dirac operator on the corresponding Clifford module is $\sqrt{2}(\nabla' + \nabla'^*)$.*

Let M be a hyperkähler manifold. Then there is a covariantly constant inclusion of $Sp(1)$ as the gauge subgroup of Clifford bundle's sections. Using a fixed complex structure I from the hyperkähler family of complex structures, we can construct a spin bundle $\mathcal{S}$ over M as above. On the hyperkähler manifold thus constructed spin bundle does not actually depend on the choice of the complex structure I. This observation may be exploited to prove the following proposition.

Proposition 7 *Let $\mathcal{V}$ be a hyperholomorphic vector bundle with hermitian metric on a hyperkähler manifold. Then for any compatible complex structure C twisted spinor bundle $\mathcal{V} \otimes \mathcal{S} = \mathcal{V} \otimes \mathcal{S}^+ \oplus \mathcal{V} \otimes \mathcal{S}^-$ is isomorphic to $\Lambda_C^{*,0}(\mathcal{V}, M) = \Lambda_C^{even,0}(\mathcal{V}, M) \oplus \Lambda_C^{odd,0}(\mathcal{V}, M)$. Under this isomorphism the Dirac operator goes into $\sqrt{2}(\nabla_C' + (\nabla_C')^*)$.*

Consider the spin bundle $\mathcal{S}$ over $Cl(M)$ as the bundle of $(*, 0)-$forms with respect to I. Then $\Omega = \frac{1}{4}(\omega^J - i\omega^K)$ is covariantly constant $(2, 0)-$form, therefore Ω is holomorphic. As a direct consequence of this fact and the Proposition 5 one can obtain, that the operator e acts as the exterior product with Ω and the operator f acts as the contraction with Ω on the space of twisted spinors. The last space is identified with the space of $(*, 0)-$forms.

References

[1] Atiyah M., Singer I., The index of elliptic operators IV, *Ann. of Math.*, 1971, 93, 119–138

[2] Atiyah M., Hitchin N., Singer I., Self-duality in four-dimensional Riemannian geometry, *Proc. R. Soc. Lond. A*, 1978, V.362, 425–461

[3] Belavin A., Fateev V., Schwarz A., Tyupkin Yu., Quantum fluctuations and multi-instanton solutions, *Phys. Lett.*, 1979, V.83B, N 3,4, 317–320

[4] Berline N., Getzler E., Vergne M., Heat kernels and Dirac operators, *Springer-Verlag, Berlin, Heidelberg*, 1992, 369pp

[5] Berkovits N., Vafa C., $N = 4$ Topological strings, *Nucl.Phys. B*, 1995, 433, 123–180

[6] Bismut J.-M., Local index theory and higher analytic torsion, *Documenta Matematica - Extra Volume ICM 1998*, 1, 143–162

[7] Bismut J.-M., Gillet H., Soulé C., Analitic torsion and holomorphic determinant bundles I, II, III, *Comm. Math. Phys.*, 1988, V.115, 49–78, 79–126, 301–351

[8] Braam P., Baal P., Nahms transformation for instantons, *Comm. Math. Phys.*, 1989, V.122, 267–280

[9] Corrigan E., Goddard P., Construction of instanton and monopole solutions and reciprocity, *Annals of Phys.*, 1984, V.154, 253–279

[10] Donaldson S., Kronheimer P., The geometry of four-manifolds, *Clarendon Press, Oxford*, 1990, 440pp

[11] Gillet H., Soulé C., Characteristic Classes for Algebraic Vector Bundles with Hermitian metric, *Annals of Math.*, 1990, 131, 163–203, 205–238.

[12] Griffits Ph., Harris J., Principles of algebraic geometry, *Wiley-Interscience, New-York*, 1978

[13] Hitchin N., Hyperkähler manifolds, *Séminaire Bourbaki*, 1991, N 748, 1–25

[14] Quillen D., Determinants of Cauchy-Riemann operators over a Riemann surface, *Funct. Anal. Appl.*, 1985, V.19, 31–34

[15] Ray D., Singer I., Analitic torsion for complex manifolds, *Ann. Math.*, 1973, V.98, N 1, 154–177

[16] Schenk H., On generalized Fourier transform of instantons over flat tori, *Comm. Math. Phys.*, 1988, V.116, 117–183

[17] Verbitsky M., Hyperholomorphic bundles over hyperkähler manifolds, *J. Alg. Geom.*, 1996, 5, 633-669